ON TRANSFINITE EXTENSION OF ASYMPTOTIC DIMENSION

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Abstract. We prove that a transfinite extension of asymptotic dimension asind is trivial. We introduce a transfinite extension of asymptotic dimension asdim and give an example of metric proper space which has transfinite infinite dimension.

0. Asymptotic dimension asdim of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups [1]. This dimension can be considered as asymptotic analogue of the Lebesgue covering dimension dim. Dranishnikov has introduced dimensions asInd and asind which are analogous to large inductive dimension Ind and small inductive dimension ind [2,3].It is known that $\operatorname{asdim} X = \operatorname{asInd} X$ for each proper metric space with $\operatorname{asdim} X < \infty$. The problem of coincidence of asdim and asInd is still open in the general case [3].

Extending codomain of Ind to ordinal numbers we obtain the transfinite extension trInd of the dimension Ind. It is known that there exists a space S_α such that $\operatorname{trInd} S_\alpha = \alpha$ for each countable ordinal number α [4]. Zarichnyi has proposed to consider transfinite extension of asInd and conjectured that this extension is trivial. It is proved in [5] that if a space has a transfinite asymptotic dimension trasInd, then this dimension is finite.

We investigate in this paper transfinite extensions for the asymptotic dimensions asind and asdim. It appears that extending codomain of asind to ordinal numbers we obtain the trivial extension as well. However, the main result of this paper is construction of transfinite extension trasdim of asdim which is not trivial. Moreover, trasdim classifies the metric spaces with asymptotic property C introduced by Dranishnikov [8].

The paper is organized as follows: in Section 1 we give some necessary definitions and introduce some denotations, in Section 2 we prove that the transfinite extension

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of asind is trivial and in Section 3 we define the transfinite extension trasdim of asdim and build a proper metric space X such that $\operatorname{trasdim} X = \omega$.

1. Let $A_1, A_2 \subset X$ be two disjoint closed subsets in a topological space X. We recall that a *partition* between A_1 and A_2 is a subset $C \subset X$ such that there are open disjoint sets U_1, U_2 satisfying the conditions: $X \setminus C = U_1 \cup U_2$, $A_1 \subset U_1$ and $A_2 \subset U_2$. Clearly a partition C is a closed subset of X.

We will define the asymptotic dimensions asind and asInd for the class of proper metric space. We recall that a metric space is *proper* if every closed ball is compact. We assume that some base point $x_0 \in X$ is chosen for each proper metric space X. The generic metric we denote by d. If X is a metric space and $A \subset X$ we denote by $B_r(A)$ the open r-neighborhood: $B_r(A) = \{x \in X \mid d(x, A) < r\}$. We call two subsets $A_1, A_2 \subset X$ in a metric space X *asymptotically disjoint* if $\lim_{r\to\infty} d(A_1 \setminus B_r(x_0), A_2 \setminus B_r(x_0)) = \infty$.

A map $\phi : X \to I = [0, 1]$ is called *slowly oscillating* if for any $r > 0$, for given $\varepsilon > 0$ there exists $D > 0$ such that $\operatorname{diam} \phi(B_r(x)) < \varepsilon$ for any x with $d(x, x_0) \geq D$. If $C_h(X)$ is the set of all continuous slow oscillating functions $\phi : X \to I$, then the *Higson compactification* is the closure of the image of X under the embedding $\Phi : X \to I^{C_h(X)}$ defined as $\Phi(x) = (\phi(x) \mid \phi \in C_h(X)) \in I^{C_h(X)}$. We denote the Higson compactification of a proper metric space X by cX and the remainder $cX \setminus X$ by νX. The compactum νX is called *Higson corona*. Let us remark that νX does not need to be metrizable.

Let C be a subset of a proper metric space X. By C' we denote the intersection $\operatorname{cl} C \cap \nu X$ of the closure $\operatorname{cl} C$ in the Higson compactification cX. Clearly, two sets A_1 and A_2 are asymptotically disjoint iff their traces A'_1 and A'_2 in the Higson corona are disjoint. Note that for each $r > 0$ we have $B_r(C)' = C'$.

Let $A_1, A_2 \subset X$ be two asymptotically disjoint subsets of a proper metric space X. A subset $C \subset X$ is called an *asymptotic separator* for A_1 and A_2 if its trace C' is a partition for A'_1 and A'_2 in νX.

We define $\operatorname{asInd} X = -1$ if and only if X is bounded; $\operatorname{asInd} X \leq n$ if for every two asymptotically disjoint sets A, $B \subset X$ there is an asymptotic separator C with $\operatorname{asInd} C \leq n - 1$. Naturally we say $\operatorname{asInd} X = n$ if $\operatorname{asInd} X \leq n$ and it is not true that $\operatorname{asInd} X \leq n - 1$. We set $\operatorname{asInd} X = \infty$ if $\operatorname{asInd} X > n$ for each $n \in \mathbb{N}$ [2].

Let $a \in \nu X$ and $A \subset X$ such that $a \notin A'$. A subset $C \subset X$ is called an

asymptotic separator for x and A if its trace C' is a partition for $\{x\}$ and A' in νX.

We define $\operatorname{asind} X = -1$ if and only if X is bounded; $\operatorname{asind} X \leq n$ if for every $x \in \nu X$ and $A \subset X$ such that $x \notin A'$ there is an asymptotic separator C with $\operatorname{asind} C \leq n-1$. Naturally we say $\operatorname{asind} X = n$ if $\operatorname{asind} X \leq n$ and it is not true that $\operatorname{asind} X \leq n-1$. We set $\operatorname{asind} X = \infty$ if $\operatorname{asind} X > n$ for each $n \in \mathbb{N}$ [3].

There are proved subspace and addition theorems for asInd in [5]:

Theorem A. *Let X be a proper metric space and $Y \subset X$. Then* $\operatorname{asInd} Y \leq \operatorname{asInd} X$.

Theorem B. *Let X be a proper metric space and $X = Y \cup Z$ where Y and Z are unbounded sets. Then* $\operatorname{asInd} X \leq \operatorname{asInd} Y + \operatorname{asInd} Z$.

Define the transfinite extension $\operatorname{trasInd} X$: $\operatorname{trasInd} X = -1$ if and only if X is bounded; $\operatorname{trasInd} X \leq \alpha$ where α is an ordinal number if for every two asymptotically disjoint sets A, $B \subset X$ there is an asymptotic separator C with $\operatorname{trasInd} C \leq \beta$ for some $\beta < \alpha$. Naturally we say $\operatorname{trasInd} X = \alpha$ if $\operatorname{trasInd} X \leq \alpha$ and it is not true that $\operatorname{trasInd} X \leq \beta$ for some $\beta < \alpha$. We set $\operatorname{trasInd} X = \infty$ if for each ordinal number α it is not true that $\operatorname{trasInd} X \leq \alpha$. It is proved in [5] that this extension is trivial:

Theorem C. *Let X be a proper metric space such that* $\operatorname{trasInd} X < \infty$. *Then* $\operatorname{asInd} X < \infty$.

2. We consider in this section a transfinite extension of asymptotic dimension asind and show that this extension is trivial.

Define the transfinite extension $\operatorname{trasind} X$: $\operatorname{trasind} X = -1$ if and only if X is bounded; $\operatorname{trasind} X \leq \alpha$ where α is an ordinal number if for every $x \in \nu X$ and $A \subset X$ such that $x \notin A'$ there is an asymptotic separator C with $\operatorname{trasind} C \leq \beta$ for some $\beta < \alpha$. Naturally we say $\operatorname{trasind} X = \alpha$ if $\operatorname{trasind} X \leq \alpha$ and it is not true that $\operatorname{trasind} X \leq \beta$ for some $\beta < \alpha$. We set $\operatorname{trasind} X = \infty$ if for each ordinal number α it is not true that $\operatorname{trasind} X \leq \alpha$. It follows from the definition that $\operatorname{asind} X < \infty$ iff $\operatorname{trasind} X < \omega$ where ω is the first infinite ordinal number.

Lemma 1. *Let* $\operatorname{trasind} X = \alpha$ *for some ordinal number α. Then for each $\beta < \alpha$ there exists a subset $X_\beta \subset X$ such that* $\operatorname{trasind} X_\beta = \beta$.

Proof. We shall apply transfinite induction with respect to α. For $\alpha = 0$ the lemma is obvious. Assume that the theorem holds for all $\alpha < \alpha_0 \geq 1$ and consider a proper metric space X such that $\operatorname{trasind} X = \alpha_0$ as well an ordinal number $\beta < \alpha_0$. Suppose that X contains no subset M with $\operatorname{trasind} M = \beta$. By the inductive assumption X contains no subset M' which satisfies $\beta \leq \operatorname{trasind} M' < \alpha_0$. Thus for every point $x \in \nu X$ and each $A \subset X$ such that $x \notin A'$ there exists an asymptotic separator C for x and A such that $\operatorname{trasind} C < \beta$. This contradicts, however, the equality $\operatorname{trasind} X = \alpha_0$, so that X contains a subset X_β with $\operatorname{trasind} X_\beta = \beta$.

It is proved in [3] that $\operatorname{asind} X \leq \operatorname{asInd} X$.

Lemma 2. *Let* $\operatorname{asind} X < \infty$ *for some proper metric space* X*. Then* $\operatorname{asInd} X < \infty$ *as well.*

Proof. We use induction with respect to $\operatorname{asind} X$. If $\operatorname{asind} X = -1$, then $\operatorname{asInd} X = -1$. Suppose we have proved the lemma for each $i < n \geq 0$. Consider any proper metric space X with $\operatorname{asind} X \leq n$. Let A and B be asymptotically disjoint subsets of X and a is any point of A'. Since $\operatorname{asind} X \leq n$, there exists an asymptotic separator L_a between a and B such that $\operatorname{asind} L_a < n$. Consider open disjoint sets U_a, V_a in νX such that $a \in U_a$, $B' \subset V_a$ and $\nu X \setminus L'_a = U_a \cup V_a$. Since A' is compact, there exist points $a_1, \dots, a_k \in A'$ such that $A' \subset \cup_{i=1}^k U_{a_i}$. Put $U = \cup_{i=1}^k U_{a_i}$, $V = \cap_{i=1}^k V_{a_i}$ and $S = \nu X \setminus (U \cup V)$. Then U and V are open disjoint subsets of νX such that $A' \subset U$ and $B' \subset V$. Hence S is a partition between A' and B' in νX. Moreover, $S = \cup_{i=1}^k L'_{a_i} \setminus U$. Choose a continuous function $f : \nu X \to [0,1]$ such that $f(A') \subset \{0\}$ and $f(\nu X \setminus U) \subset \{1\}$. We can extend this function to a continuous function $F : cX \to [0,1]$. Put $L = (\cup_{i=1}^k L_{a_i}) \setminus (F^{-1}([0,\frac{1}{2}]) \cap X)$. Then we have $S \subset L'$ and hence L is an asymptotic separator between A and B.

Since $\operatorname{asind} L_{a_i} < n$, we have $\operatorname{asInd} L_{a_i} < \infty$ for each i by inductive assumption. Hence we have $\operatorname{asInd} L \leq \operatorname{asInd} \cup_{i=1}^k L_{a_i} < \infty$ by Theorems A and B. So, $\operatorname{trasInd} X \leq \omega$ and $\operatorname{asInd} X < \infty$ by Theorem C. The lemma is proved

Theorem 1. *Let* $\operatorname{trasind} X < \infty$ *for some proper metric space* X*. Then* $\operatorname{asind} X < \infty$ *as well.*

Proof. Suppose the contrary. Then there exists a proper metric space X such that $\operatorname{trasind} X = \alpha$ for some ordinal number $\alpha \geq \omega$. We can choose a proper metric

space Y such that $\operatorname{trasind} Y = \omega$. Let us show that $\operatorname{asInd} Y < \infty$. Choose any asymptotically disjoint sets A and B in Y. Since $\operatorname{trasind} Y = \omega$, we can choose for each point $a \in A'$ an asymptotic separator L_a between a and B such that $\operatorname{asind} L_a < \infty$. So, $\operatorname{asInd} L_a < \infty$ by Lemma 2. Using the same method as in the proof of Lemma 2, we can choose an asymptotic separator L between A and B such that $\operatorname{asInd} L < \infty$. Hence, $\operatorname{trasInd} Y \le \omega$ and $\operatorname{asInd} Y < \infty$ by Theorem C. Then $\operatorname{asind} Y \le \operatorname{asInd} Y < \infty$ and we obtain the contradiction. The theorem is proved.

3. In this section we introduce a transfinite extension of dimension asdim introduced by Gromov [1]. A family $\mathcal{A}$ of subsets of a metric space is called *uniformly bounded* if there exists a number $C > 0$ such that $\operatorname{diam} A \le C$ for each $A \in \mathcal{A}$; $\mathcal{A}$ is called *r-disjoint* for some $r > 0$ if $d(A_1, A_2) \ge r$ for each A_1, $A_2 \in \mathcal{A}$ such that $A_1 \ne A_2$.

The *asymptotic dimension* of a metric space X does not exceed $n \in \mathbb{N} \cup \{0\}$ (written $\operatorname{asdim} X \le n$) iff for every $D > 0$ there exists a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_n$, where all $\mathcal{U}_i$ are D-disjoint. Moreover, we put $\operatorname{asdim} X = -1$ iff X is bounded.

Since the definition of asdim is not inductive, we cannot immediately extend this dimension. We need some set-theoretical construction used by Borst to extend covering dimension and metric dimension [6,7].

Let L be an arbitrary set. By $FinL$ we shall denote the collection of all finite, non-empty subsets of L. Let M be a subset of $FinL$. For $\sigma \in \{\emptyset\} \cup FinL$ we put

$$M^\sigma = \{\tau \in FinL \mid \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

Let M^a abbreviate $M^{\{a\}}$ for $a \in L$.

Define the ordinal number $\operatorname{Ord} M$ inductively as follows

$\operatorname{Ord} M = 0$ iff $M = \emptyset$,

$\operatorname{Ord} M \le \alpha$ iff for every $a \in L$, $\operatorname{Ord} M^a < \alpha$,

$\operatorname{Ord} M = \alpha$ iff $\operatorname{Ord} M \le \alpha$ and $\operatorname{Ord} M < \alpha$ is not true, and

$\operatorname{Ord} M = \infty$ iff $\operatorname{Ord} M > \alpha$ for every ordinal number α.

We will need some lemmas from [6]:

Lemma D. *Let L be a set and let M be a subset of $FinL$. In addition let $n \in \mathbb{N}$. Then $\operatorname{Ord} M \le n$ iff $|\sigma| \le n$ for each $\sigma \in M$.*

We call a subset M of $FinL$ *inclusive iff for every* σ, $\sigma' \in FinL$ *such that* $\sigma \in M$ *and* $\sigma' \subset \sigma$ *also* $\sigma' \in M$.

Lemma E. *Let L be a set and let M be an inclusive subset of $FinL$. Then* $\operatorname{Ord} M = \infty$ *iff there exists a sequence $(a_i)_{i=1}^{\infty}$ of distinct elements of L such that $\sigma_n = \{a_i\}_{i=1}^n \in M$ for each $n \in \mathbb{N}$.*

Lemma F. *Let $\phi : L \to L'$ be a function and let $M \subset FinL$ and $M' \subset FinL'$ be such that for every $\sigma \in M$ we have $\phi(\sigma) \in M'$ and $|\phi(\sigma)| = |\sigma|$. Then* $\operatorname{Ord} M \le \operatorname{Ord} M'$.

Let us define the following collection for a metric space (X, d):

$$A(X,d) = \{\sigma \in Fin\mathbb{N} \mid \text{ there is no uniformly bounded families } \mathcal{V}_i \text{ for } i \in \sigma$$

$$\text{such that } \cup_{i\in\sigma} \mathcal{V}_i \text{ covers } X \text{ and } \mathcal{V}_i \text{ is } i-\text{disjoint}\}.$$

Let (X, d) be a metric space. Then put $\operatorname{trasdim} X = \operatorname{Ord} A(X, d)$ and $\operatorname{trasdim} X = -1$ iff X is bounded. It follows from Lemma D that trasdim is a transfinite extension of asdim: $\operatorname{trasdim} X \le n$ iff $\operatorname{asdim} X \le n$ for each $n \in \mathbb{N}$.

Dranishnikov has defined asymptotic property C as follows: a metric space X has asymptotic property C if for any sequence of natural numbers $n_1 < n_2 < \dots$ there is a finite sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^n$ such that $\cup_{i=1}^n \mathcal{U}_i$ covers X and $\mathcal{U}_i$ is n_i-disjoint [8].

The next proposition follows from Lemma E:

Proposition 1. *A metric space X has asymptotic property C iff* $\operatorname{trasdim} X < \infty$.

Proposition 2. *Let X be a metric space and $Y \subset X$. Then* $\operatorname{trasdim} Y \le \operatorname{trasdim} X$.

Proof. Put $M = A(Y)$, $M' = A(X)$ and $\phi = \mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$. Then M, M' and ϕ satisfy the condition of Lemma F and $\operatorname{trasdim} Y = \operatorname{Ord} A(Y) \le \operatorname{Ord} A(X) = \operatorname{trasdim} X$.

We are going to construct two examples: a proper metric space L_ω such that $\operatorname{trasdim} L_\omega = \omega$ which shows that this extension is not trivial and a proper metric space L_∞ such that $\operatorname{trasdim} L_\infty = \infty$.

We denote by $N_R(A) = \{x \in X | d(x, A) \le R\}$ for a metric space X, $A \subset X$ and $R > 0$.

We consider $\mathbb{Z}^n$ with the sup-metric defined as follows $d((k_1,\dots,k_n),(l_1,\dots,l_n)) = \max\{|k_1-l_1|,\dots,|k_n-l_n|\}$. It follows from [1,Lemma 6.1] that $\operatorname{asdim}\mathbb{Z}^n \le n$. By $k\mathbb{Z}$ we denote $\{kl|l\in\mathbb{Z}\}$ for some $k\in\mathbb{N}$.

Lemma 3. *There exist no uniformly bounded $2k$-disjoint families $\mathcal{V}_1,\dots,\mathcal{V}_n$ in $(k\mathbb{Z})^n$ such that $\cup_{i=1}^n\mathcal{V}_i$ covers $(k\mathbb{Z})^n$.*

Proof. Suppose the contrary. Then there exists $s\in\mathbb{N}$ such that $\operatorname{diam} V\le s$ for each $i\in\{1,\dots,n\}$ and $V\in\mathcal{V}_i$. We can suppose that $s\ge 2k$. Consider $(k\mathbb{Z})^n$ as the subset of $\mathbb{R}^n$. Put $\mathcal{V}=\cup_{i=1}^n\mathcal{V}_i$ and $\mathcal{U}=\{N_{k/2}(V)\cap[-s,s]^n|V\in\mathcal{V}\}$. Then $\mathcal{U}$ is a finite closed cover of the cube $[-s,s]^n$ no member of which meets two opposite faces of $[-s,s]^n$ and each subfamily of $\mathcal{U}$ containing $n+1$ distinct elements of $\mathcal{U}$ has empty intersection. We obtain the contradiction with the Lebesgue's Covering Theorem [4, Theorem 1.8.20].

Corollary. $\operatorname{trasdim}(k\mathbb{Z})^n = \operatorname{asdim}(k\mathbb{Z})^n = n$ *for each* $k,n\in\mathbb{N}$.

Put $X=\cup_{i=1}^\infty\mathbb{Z}^i$. Define a metric in X. Let $a=(a_1,\dots,a_l)\in\mathbb{Z}^l$ and $b=(b_1,\dots,b_k)\in\mathbb{Z}^k$. Suppose that $l\le k$. Consider $a'=(a_1,\dots,a_l,0,\dots,0)\in\mathbb{Z}^k$. Put $c=0$ if $l=k$ and $c=l+(l+1)+(l+2)+\dots+(k-1)$ if $l<k$. Now, define $d_\infty(a,b)=\max\{d(a',b),c\}$ where d is sup-metric in $\mathbb{Z}^k$.

Consider the proper metric space $L_\infty=(X,d_\infty)$ and its subspace $L_\omega=\cup_{k=1}^\infty(k\mathbb{Z})^k\subset L_\infty$

Lemma 4. *Let Y be a metric space and $X\subset Y$. Then* $\operatorname{trasdim} N_n(X)=\operatorname{trasdim} X$ *for each $n\in\mathbb{N}$.*

Proof. Consider the function $\phi:\mathbb{N}\to\mathbb{N}$ defined as follows $\phi(k)=k+2n$ for $k\in\mathbb{N}$. Obviously, we have $|\phi(\tau)|=|\tau|$ for each $\tau\in Fin\mathbb{N}$. Consider any $\tau\in\{k_1,\dots,k_l\}\in A(N_n(X))$. Suppose that $\phi(\tau)=\{k_1+2n,\dots,k_l+2n\}\notin A(X)$. Then there exist a sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^l$ such that $\cup_{i=1}^l\mathcal{U}_i$ covers X and $\mathcal{U}_i$ is k_i+2n-disjoint for each $i\in\{1,\dots,l\}$. Consider the family $N_n(\mathcal{U}_i)=\{N_n(V)|V\in\mathcal{U}_i\}$ for each $i\in\{1,\dots,l\}$. Then the families $N_n(\mathcal{U}_1),\dots,N_n(\mathcal{U}_l)$ are uniformly bounded, $\cup_{i=1}^l N_n(U_i)$ covers $N_n(X)$ and $\mathcal{U}_i$ is k_i-disjoint for each $i\in\{1,\dots,l\}$. We obtain the contradiction. So, $\phi(\tau)\in A(X)$ and $\operatorname{trasdim} N_n(X)\le\operatorname{trasdim} X$ by Lemma F. The inequality $\operatorname{trasdim} N_n(X)\ge\operatorname{trasdim} X$ follows from Proposition 2.

Theorem 2. $\operatorname{trasdim} L_\omega = \omega$.

Proof. The inequality $\operatorname{trasdim} L_\omega \geq \omega$ follows from Proposition 2 and Corollary.

Consider any $n \in \mathbb{N}$. Let us show that $\operatorname{Ord} A(L_\omega)^n \leq n-1$. Consider any $\tau = \{k_1, \dots, k_n\} \in Fin\mathbb{N}$ such that $n \notin \tau$. It is enough to show that $\tau \cup \{n\} \notin A(L_\omega)$.

Since $\operatorname{asdim}(n\mathbb{Z})^n$ and $\cup_{i=1}^n (i\mathbb{Z})^i \subset N_{1+2+\dots+n-1}(n\mathbb{Z}^n)$, we have that $\operatorname{trasdim} \cup_{i=1}^n (i\mathbb{Z})^i \leq n$. Then there exist a sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^{n+1}$ such that $\cup_{i=1}^{n+1}\mathcal{U}_i$ covers $\cup_{i=1}^n (i\mathbb{Z})^i$, $\mathcal{U}_i$ is k_i-disjoint for each $i \in \{1, \dots, n\}$ and $\mathcal{U}_{n+1}$ is n-disjoint. Consider the family $\mathcal{V} = \mathcal{U}_{n+1} \cup \{\{x\} | x \in \cup_{i=n+1}^\infty (i\mathbb{Z})^i\}$. Then $\mathcal{V}$ is n-disjoint uniformly bounded family such that $(\cup_{i=1}^n \mathcal{U}_i) \cup \mathcal{V}$ covers L_ω. Hence $\tau \cup \{n\} \notin A(L_\omega)$.

Theorem 3. $\operatorname{trasdim} L_\infty = \infty$.

Proof. Suppose the contrary. Consider the sequence $(n_i)_{i=1}^\infty$ where $n_i = i+1$. Then there exists $k \in \mathbb{N}$ such that $\{2, 3, \dots, k+1\} \notin A(L_\infty)$. But then $\{2, 3, \dots, k+1\} \notin A(\mathbb{Z}^k)$ and we obtain the contradiction with Lemma 3.

Finally, we will prove that trasdim could have only countable values.

Lemma 5. *Let X be a metric space and $\tau \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\operatorname{Ord} A(X)^\tau = \alpha$ for some ordinal number α. Then for each $\xi \leq \alpha$ there exists $\sigma \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\operatorname{Ord} A(X)^{\tau\cup\sigma} = \xi$.*

Proof. We shall apply the transfinite induction with respect to α. For $\alpha = 0$ the lemma is obvious. Assume that the lemma holds for all $\alpha < \alpha_0$ and consider a metric space X and $\tau \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\operatorname{Ord} A(X)^\tau = \alpha_0$ as well as an ordinal number $\xi < \alpha_0$. Suppose that there is no $\sigma \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\operatorname{Ord} A(X)^{\tau\cup\sigma} = \xi$. By the inductive assumption there is no $\sigma' \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\xi \leq \operatorname{Ord} A(X)^{\tau\cup\sigma'} < \alpha_0$. Then $\operatorname{Ord} A(X)^{\tau\cup\{n\}} < \xi$ for each $n \in \mathbb{N} \setminus \tau$ and we obtain the contradiction with $\operatorname{Ord} A(X)^\tau = \alpha_0$.

Theorem 4. *If we have $\operatorname{trasdim} X < \infty$ for some metric space X, then $\operatorname{trasdim} X < \omega_1$.*

Proof. Suppose the contrary. Then there exists a metric space Xsuch that $\operatorname{trasdim} X \geq \omega_1$. We can choose $\tau \in Fin\mathbb{N} \cup \{\emptyset\}$ such that $\operatorname{Ord} A(X)^\tau = \omega_1$. Then for each $n \in \mathbb{N} \setminus \tau$ we have $\operatorname{Ord}(A(X)^\tau)^n = \xi_n < \omega_1$. Then $\omega_1 = \sup\{\xi_n | n \in \mathbb{N} \setminus \tau\}$ and we obtain the contradiction.

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